

Large N and Bethe ansatz

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Abstract

We describe an integrable model, related to the Gaudin magnet, and its relation to the matrix model of Brézin, Itzykson, Parisi and Zuber. Relation is based on Bethe ansatz for the integrable model and its interpretation using orthogonal polynomials and saddle point approximation. Large N limit of the matrix model corresponds to the thermodynamic limit of the integrable system. In this limit (functional) Bethe ansatz is the same as the generating function for correlators of the matrix models.

1. Introduction

Matrix models in the large N -limit have been studied from different points of view since the seminal papers of t'Hooft [1] and Brézin, Itzykson, Parisi and Zuber [2]. There is no doubt of their importance in 2D gravity and string theory (see e.g. [3], [4], [5] and [6] for reviews on their different aspects). Importance of CFT and of integrable systems for matrix models have also been known for some time (see e.g. [4], [5]). Most recently there is a great interest in integrable spin chains and Bethe ansatz in relation with large N -limit of $\mathcal{N} = 4$ SYM (see e.g. [7], [8], [9] and references therein). This possibly may help to understand more precisely the large N conjecture of Maldacena [10].

The purpose of this paper is to make precise the relation between the large N -limit of the one-matrix model and Bethe ansatz for some integrable model related to the classical rational r -matrix (Gaudin type model). This could provide a toy model of the above mentioned relation between the Heisenberg model and $\mathcal{N} = 4$ SYM.

We start with a very short review of the one-matrix model following [3]. Two basic methods of its solution, the saddle point approximation and the method of orthogonal polynomials, contain implicitly all information about some particular integrable model. Then we recall the version of Bethe ansatz appropriate for models related to classical r -matrices [11], [12], [13]. This can be viewed as some specific limit [11] of the quantum R -matrix Bethe ansatz [14], [11], [15], [16] which is appropriate for Heisenberg chain type of models. Finally we will go on with an explicit description of the relevant integrable model and discuss algebraic as well as functional Bethe ansätze for it. We will relate in a precise way the large N -limit of the matrix model and the thermodynamic limit of our integrable system. In this limit Bethe eigenfunction give the generating function for correlators of the matrix model. Also the norm of the Bethe eigenstate is nicely related to the large N expansion the partition function Z_N .

2. Matrix model of BIPZ

One-matrix model is treated in detail in many places [3], [4]. So we can be very brief. The task is to perform the integral over $N \times N$ hermitian matrix M

$$Z_N = \int d^{N^2} M \exp [-(N/g) \text{Tr } V(M)], \quad (1)$$

where $V(M)$ is a general polynomial potential $V(M) = M^2 + \sum_{k \geq 3} \beta_k \lambda^k$ and the integration is over N^2 independent real variables which are the real and imaginary parts of matrix elements of M . The standard strategy is to diagonalize the matrix M , i.e. write $M = U \text{diag}(\lambda_1, \dots, \lambda_N) U^\dagger$ with an unitary U . Noticing that the integrand depends only on N eigenvalues λ_i and integrating out the unitary group one gets

$$Z_N = \int \prod_{i=1}^N d\lambda_i \exp \left[-(N/g) \sum_i V(\lambda_i) + \sum_{i \neq j} \ln(\lambda_i - \lambda_j) \right]. \quad (2)$$

The logarithmic part in the exponent justifies the saddle point approximation for $N \rightarrow \infty$. This gives the configuration with the minimal energy

$$\frac{2g}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = V'(\lambda_i). \quad (3)$$

In order to solve equation (3), it is convenient to introduce the resolvent function

$$W(z) = \frac{1}{N} \sum_i \left\langle \frac{1}{z - \lambda_i} \right\rangle. \quad (4)$$

One also introduces the averaged density of eigenvalues $\rho(\lambda)$

$$\rho(\lambda) = \frac{1}{N} \sum_i \langle \delta(\lambda - \lambda_i) \rangle. \quad (5)$$

If we assume for simplicity that the potential V has only one well (or at least one deepest well), then the support of $\rho(\lambda)$ will be in some interval $[a, b]$ (in general it can be a collection of disjoint intervals). We have relations

$$W(z) = \int_a^b d\lambda \frac{\rho(\lambda)}{z - \lambda} \quad (6)$$

and

$$\rho(\lambda) = -\frac{1}{2\pi i} (W(\lambda + i0) - W(\lambda - i0)). \quad (7)$$

The solution for $W(z)$ is

$$W(z) = \frac{1}{4\pi i} \oint dx \frac{V'(x)}{z - x} \frac{\sqrt{(z - a)(z - b)}}{\sqrt{(x - a)(x - b)}}, \quad (8)$$

where the integration contour encloses the cut $[a, b]$. Finally the solution to (3) is given by the density (7). See the above cited references and [17] for more details concerning the solution of this equation.

An alternative method of solving (1) is to use orthogonal polynomials $P_n(\lambda) = \lambda^n + \dots$ with respect to the measure

$$\exp[-(N/g)V(\lambda)]d\lambda. \quad (9)$$

Partition function Z_N is then simply expressed in terms of their norms squared $s_n = \langle P_n | P_n \rangle$ as

$$Z_N = N! s_0 \dots s_{N-1}. \quad (10)$$

This follows easily from the integral representation for P_n which will be given later (37), (38). See again [3], [4], [17] for a more explicit method of determination of these polynomials. This is based on recurrence relations between orthogonal polynomials which can be solved in the large N -limit. This little knowledge about the one-matrix model is enough to understand it as an quantum integrable model when N is large.

3. Bethe ansatz

Before describing our integrable model we give a brief review of the relevant Bethe ansatz following [13]. Let $L(\lambda)$ be a 2×2 matrix

$$\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix} \quad (11)$$

its matrix elements being operators depending on complex parameter λ , acting in some Hilbert space \mathcal{H} . We assume the following form of the commutation relations for the matrix elements of $L(\lambda)$

$$[L(\lambda) \otimes I, I \otimes L(\mu)] + [r(\lambda - \mu), L(\lambda) \otimes I + I \otimes L(\mu)] = 0, \quad (12)$$

where I is the unit 2×2 matrix and $r(\lambda - \mu)$ is the classical r -matrix

$$\frac{1}{\lambda} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (13)$$

Note that the commutator in (12) is a matrix commutator respecting the operator nature of matrix elements of $L(\lambda)$. Further, we put

$$T(\lambda) = \frac{1}{2} \text{Tr } L(\lambda)^2, \quad (14)$$

where Tr is the “ 2×2 matrix trace”. as a consequence of (12) we have the following useful commutation relations

$$[A(\lambda), B(\mu)] = -\frac{1}{\lambda - \mu} B(\mu) + \frac{1}{\lambda - \mu} B(\lambda), \quad (15)$$

$$[B(\lambda), B(\mu)] = 0, \quad (16)$$

$$[B(\lambda), C(\lambda)] = 2A'(\lambda) \quad (17)$$

and

$$\begin{aligned} [T(\lambda), B(\mu)] &= \frac{1}{\lambda - \mu} A(\mu) B(\lambda) - \frac{1}{\lambda - \mu} B(\mu) A(\lambda) \\ &\quad - \frac{1}{\lambda - \mu} A(\lambda) B(\mu) + \frac{1}{\lambda - \mu} B(\lambda) A(\mu). \end{aligned} \quad (18)$$

Operator $T(\lambda)$ is the generating function of integrals of motion in involution as

$$[T(\lambda), T(\mu)] = 0. \quad (19)$$

Next we assume that there is the so-called pseudo-vacuum $|0\rangle \in \mathcal{H}$ such that

$$C(\lambda)|0\rangle = 0, \quad (20)$$

$$A(\lambda)|0\rangle = a(\lambda)|0\rangle \quad (21)$$

and hence

$$T(\lambda)|0\rangle = (a(\lambda)^2 + a'(\lambda)) |0\rangle \quad (22)$$

holds true. Algebraic Bethe ansatz (ABA) is then given by a family of vectors

$$|\lambda_1, \dots, \lambda_N\rangle = B(\lambda_1) \dots B(\lambda_N) |0\rangle. \quad (23)$$

These are, as easy verified, with help of (12–18), eigenstates of $T(\lambda)$ with eigenvalues $\tau(\lambda)$

$$\tau(\lambda) = -2 \sum_i \frac{a(\lambda)}{\lambda - \lambda_i} + \sum_{i \neq j} \frac{1}{\lambda - \lambda_i} \frac{1}{\lambda - \lambda_j} + (a(\lambda)^2 + a'(\lambda)) \quad (24)$$

if and only if the numbers $\{\lambda_1, \dots, \lambda_N\}$ satisfy Bethe conditions

$$\frac{1}{2}\varphi_i \equiv a(\lambda_i) - \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = 0. \quad (25)$$

The norm squared is equal to

$$\langle \lambda_1, \dots, \lambda_N | \lambda_1, \dots, \lambda_N \rangle = \det \left| \frac{\partial \varphi_i}{\partial \lambda_j} \right|. \quad (26)$$

If we introduce the polynomial $q(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_N)$ equation (25) can be rewritten as differential equation

$$q'' - 2aq' + (a^2 - a')q = \tau q. \quad (27)$$

4. The model and its relation to BIPZ at large N

Comparing Bethe conditions (25) and the minimal energy condition (3) we see the first hint that there might be an integrable model hidden in the large N -limit of the matrix model. To describe this integrable system we need to give a representation of the algebra of (12). Let us fix an arbitrary integer K and consider the space of functions of K variables $\{x_{-1}, \dots, x_{-K}\}$. Our operators $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ will be polynomials of orders K , $K - 1$ and $K - 1$, respectively, acting on this space

$$A(\lambda) = \sum_{n=-1}^{-K-1} A_n \lambda^{-n-1}, \quad B(\lambda) = \sum_{n=-1}^{-K} B_n \lambda^{-n-1}, \quad C(\lambda) = \sum_{n=-1}^{-K} C_n \lambda^{-n-1}, \quad (28)$$

with coefficients

$$A_n = -x_m \partial_{m-n} + a_n, \quad (29)$$

$$B_n = x_n, \quad (30)$$

$$C_n = -x_m \partial_{m-n-l} \partial_l + 2a_m \partial_{m-n}. \quad (31)$$

Here the summation rule is assumed and sums run over all integers from $-\infty$ to K , assuming $\partial_n = 0$ and $x_n = 0$ whenever $n > -1$ and $\{a_{-1}, \dots, a_{-K-1}\}$ are arbitrary real constants. Note that $A_{-K-1} = a_{-K-1}$: we will assume this to be nonzero. What we have here is actually the free field representation of a factor algebra of $\widehat{sl(2)}$ with $c = 0$. ABA is applied directly to our model. Obviously pseudo-vacuum is the constant function $|0\rangle = 1$ and we have

$$a(\lambda) = \sum_{n=-1}^{-K-1} a_n \lambda^{-n-1}. \quad (32)$$

To our integrable model also Sklyanin's functional Bethe ansatz (FBA) is applicable. FBA in its full generality doesn't need a pseudo-vacuum. In our case it gives nicely the separation of variables. We will not repeat here the

general discussion of [12] and just state the result. Since $B(\lambda)$ is a polynomial of order $K - 1$,

$$B(\lambda) = x_{-K}(\lambda - u_1) \dots (\lambda - u_{K-1}), \quad (33)$$

x_{-K} and its zeros $\{u_1, \dots, u_{K-1}\}$ are independent as functions of variables $\{x_{-1}, \dots, x_{-K}\}$. We perform the change of variables

$$\{x_{-1}, \dots, x_{-K}\} \mapsto \{u_1, \dots, u_{K-1}, x_{-K}\}. \quad (34)$$

If $q(u) = (u - \lambda_1) \dots (u - \lambda_N)$ is a polynomial solution of order N to (27) then the product

$$\psi(u_1, \dots, u_{K-1}) = x_{-K}^N q(u_1) \dots q(u_{K-1}) \quad (35)$$

is an eigenfunction of $T(\lambda)$ and the corresponding eigenvalue is given by (24). The relation between the two version of Bethe ansatz is pretty obvious in our case. We go from (23) to (35) as follows

$$\begin{aligned} |\lambda_1, \dots, \lambda_N\rangle &= B(\lambda_1) \dots B(\lambda_N).1 = x_{-K}^N \prod_{i=1}^N \prod_{j=1}^{K-1} (\lambda_i - u_j) \\ &= x_{-K}^N \prod_{j=1}^{K-1} \prod_{i=1}^N (\lambda_i - u_j) = (-1)^{N(K-1)} x_{-K}^N \prod_{j=1}^{K-1} q(u_j) \\ &= (-1)^{N(K-1)} x_{-K}^N \psi(u_1, \dots, u_{K-1}). \end{aligned} \quad (36)$$

In both cases (algebraic or functional) the Bethe condition (25) just makes sure that poles of $\tau(\lambda)$ in points $\{\lambda_1, \dots, \lambda_N\}$ vanish. This must happen as the operator $T(\lambda)$ doesn't have poles in these points. If we put $a_{-1} = 0$, $a_{-2} = N/g$ and $a_{-n} = \frac{nN}{2g} \beta_n$ for $n \geq 3$, so that we have $a = \frac{N}{2g} V'$, then Bethe conditions (25) and the minimum energy condition (3) are identical.

We can go even further. Following Szegő's book on orthogonal polynomials (and being inspired by matrix model) we consider a system of orthonormal polynomials $p_N(u)$ with respect to the measure (9) given as

$$p_n(u) = \frac{(D_{n-1} D_n)^{-1/2}}{n!} \times \int \prod_{i=1}^n d\alpha_i (u - \alpha_1) \dots (u - \alpha_n) \exp \left[- \sum_i (N/g) V(\alpha_i) + \sum_{i \neq j} \ln(\alpha_i - \alpha_j) \right], \quad (37)$$

where

$$D_n = \frac{1}{(n+1)!} \int \prod_{i=1}^{n+1} d\alpha_i \exp \left[- \sum_i (N/g) V(\alpha_i) + \sum_{i \neq j} \ln(\alpha_i - \alpha_j) \right]. \quad (38)$$

Obviously we do have

$$P_n(u) = \left[\frac{D_n}{D_{n-1}} \right]^{1/2} p_n(u) \quad (39)$$

and (10) follows trivially. Let us now assume $N \rightarrow \infty$. Again the logarithmic term in the potential allows to apply the saddle point method to the integrals

defining $P_N(u)$, which are generating functions of correlators for matrix model. This is done straightforwardly with the result

$$P_N(u) = (u - \lambda_1) \dots (u - \lambda_N) + O([N/g])^{-1}. \quad (40)$$

Here $\{\lambda_1, \dots, \lambda_N\}$ must again minimize the energy, hence they satisfy (3). Recalling the functional Bethe ansatz (35) we see that the polynomials

$$P_N(u_1) \dots P_N(u_{K-1}) \rightarrow q(u_1) \dots q(u_{K-1}) \quad (41)$$

in the large N -limit and apart from an inessential factor x_{-K}^N they give Bethe eigenvectors $\psi(u_1, \dots, u_{K-1})$ of $T(\lambda)$. Also (see (26))

$$Z_N = \left[\frac{2\pi}{N/g} \right]^{N/2} \langle \lambda_1 \dots \lambda_N | \lambda_1 \dots \lambda_N \rangle^{-1/2} \times \exp \left[-(N/g) \sum_i V(\lambda_i) + \sum_{i \neq j} \ln(\lambda_i - \lambda_j) \right] (1 + O(g/N)). \quad (42)$$

Although Bethe states (23) or (35) are eigenstates of our integrable model for *any* N here we recovered them only in the large N -limit. For our representation labels $\{a_{-2}, \dots, a_{-K-1}\}$ this means that they grow proportionally to N as this goes to infinity. At the same time N is the number of vacuum excitations. Hence we can interpret the large N -limit as the thermodynamic limit of our integrable system. Clearly in this limit the polynomial $q(u)$ is the *generating function* of correlators for the matrix model.

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